UNIVERSITY OF OXFORD

PMT resources • tuition • courses

Imperial College London

MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON

Wednesday 4 November 2015

Time Allowed: 2½ hours

1449925388

Please complete the following details in BLOCK CAPITALS

Surname	
Other names	

Candidate Number	M						

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

A: Oxford Applicants: if you are applying to Oxford for the degree course:

- Mathematics *or* Mathematics & Philosophy *or* Mathematics & Statistics, you should attempt Questions **1,2,3,4,5.**
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6.**
- Computer Science *or* Computer Science & Philosophy, you should attempt **1,2,5,6,7.**

Directions under A take priority over any directions in B which are relevant to you.

B: Imperial Applicants: if you are applying to Imperial College for any of the Mathematics courses**:** Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5.**

Further credit cannot be obtained by attempting extra questions. **Calculators are not permitted.**

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

MATHEMATICS ADMISSIONS TEST

Wednesday 5 November 2014

Time Allowed: 2½ hours

Please complete these details below in block capitals.

Centre Number														
Candidate Number	**M**													
UCAS Number (if known)				–				–						

	d	d		m	m		y	y
Date of Birth			–			–		

Please tick the appropriate box:

- ☐ I have attempted Questions **1,2,3,4,5**
- ☐ I have attempted Questions **1,2,3,5,6**
- ☐ I have attempted Questions **1,2,5,6,7**

Admissions Testing Service

Administered on behalf of the University of Oxford by the Admissions Testing Service, part of Cambridge Assessment, a non-teaching department of the University of Cambridge.

FOR OFFICE USE ONLY

Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS.**

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

A.

Pick a whole number.
Add one.
Square the answer.
Multiply the answer by four.
Subtract three.

Which of the following statements are true regardless of which starting number is chosen?

I	The final answer is odd.
II	The final answer is one more than a multiple of three.
III	The final answer is one more than a multiple of eight.
IV	The final answer is not prime.
V	The final answer is not one less than a multiple of three.

(a) **I**, **II**, and **V**, (b) **I** and **IV**, (c) **II** and **V**,
(d) **I**, **III**, and **V**, (e) ✓ **I** and **V**.

Let x be the whole number

$4(x+1)^2-3 = 4(x^2+2x+1)-3$
$= 4x^2+8x+4-3$
$= 4x^2+8x+1$
$= 2(2x^2+4x)+1$

$\therefore$ always odd, I is true

if $x=1$, final answer is 13. 13 is prime so IV is false. 13 is not 1 more than a multiple of 8, so III is false.

if $x=2$, final answer is 33. 33 is not 1 more than a multiple of 3, so II is false

B. Let
$$f(x) = (x+a)^n$$
where a is a real number and n is a positive whole number, and $n \geqslant 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will

(a) always be odd, (b) ✓ always be even, (c) depend on a but not n,
(d) depend on n but not a, (e) depend on both a and n.

$f(x) = (x+a)^n$

$f'(x) = n(x+a)^{n-1}$

$f(x)$ and $f'(x)$ intersect when:

$(x+a)^n = n(x+a)^{n-1}$

$(x+a)^{n-1}[(x+a)-n] = 0$

$(x+a)^{n-1} = 0$ $\quad x = n-a$

$x = -a$

$\therefore$ there are two intersections, at $x = n-a$, $x = -a$

C. Which of the following are true for all real values of x? All arguments are in radians.

$$\begin{array}{ll}
\textbf{I} & \sin\left(\frac{\pi}{2}+x\right)=\cos\left(\frac{\pi}{2}-x\right) \\
\textbf{II} & 2+2\sin(x)-\cos^2(x)\geqslant 0 \\
\textbf{III} & \sin\left(x+\frac{3\pi}{2}\right)=\cos(\pi-x) \\
\textbf{IV} & \sin(x)\cos(x)\leqslant\frac{1}{4}
\end{array}$$

(a) **I** and **II**, (b) **I** and **III**, ✓(c) **II** and **III**,
(d) **III** and **IV**, (e) **II** and **IV**.

I $\sin\left(\frac{\pi}{2}+x\right)=\cos x$

$\cos\left(\frac{\pi}{2}-x\right)=\sin(x)$

$\cos(x)\neq\sin(x)$

$\therefore$ I is false

II $2+2\sin(x)-\cos^2(x)$

$=2+2\sin(x)-(1-\sin^2(x))$

$=1+2\sin(x)+\sin^2(x)$

$=(\sin^2(x)+1)^2\geqslant 0$ $\therefore$ II is true

III. $\cos(\pi-x)=\cos(\pi+x)$

(as $\cos(x)=\cos(-x)$)

$\sin\left(x+\frac{3\pi}{2}\right)=\sin\left(x+\frac{\pi}{2}+\pi\right)$

$\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$

$\sin\left(x+\frac{3\pi}{2}\right)=\cos(x+\pi)$

$\therefore$ III is true

IV. $\sin x\cos x=\frac{1}{2}\sin 2x$

$-1\leqslant\sin 2x\leqslant 1$

$-\frac{1}{2}\leqslant\frac{1}{2}\sin 2x\leqslant\frac{1}{2}$

$\frac{1}{2}>\frac{1}{4}$

$\therefore$ IV is false

D. Let

$$f(x)=\int_0^1(xt)^2\mathrm{d}t,\qquad\text{and}\qquad g(x)=\int_0^x t^2\mathrm{d}t.$$

Let $A>0$. Which of the following statements is true?

(a) $g(f(A))$ is always bigger than $f(g(A))$.

✓(b) $f(g(A))$ is always bigger than $g(f(A))$.

(c) They are always equal.

(d) $f(g(A))$ is bigger if $A<1$, and $g(f(A))$ is bigger if $A>1$.

(e) $g(f(A))$ is bigger if $A<1$ and $f(g(A))$ is bigger if $A>1$.

$f(x)=\int_0^1(xt)^2\,dt$

$=\int_0^1 x^2t^2\,dt$

$=\left[\frac{x^2t^3}{3}\right]_0^1$

$=\frac{x^2}{3}-0$

$g(x)=\int_0^x t^2\,dt$

$=\left[\frac{t^3}{3}\right]_0^x$

$=\frac{x^3}{3}-0$

$g(f(A))=\left(\frac{A^2}{3}\right)^3\times\frac{1}{3}$

$=\frac{A^6}{81}$

$f(g(A))=\left(\frac{A^3}{3}\right)^2\times\frac{1}{3}$

$=\frac{A^6}{27}$

$\frac{A^6}{27}>\frac{A^6}{81}$, so $f(g(A))>g(f(A))$

E. In the interval $0 \leqslant x \leqslant 2\pi$, the equation

$$\sin(2\cos(2x)+2)=0$$

has exactly

(a) 2 solutions, (b) 3 solutions, (c) 4 solutions, (d) 6 solutions, (e) 8 solutions.

$\sin(2\cos(2x)+2)=0$

let $y = 2\cos(2x) - 2$

$\sin y = 0$

$y = 0, \pi, -\pi, 2\pi, -2\pi$

$-1 \leq \cos(2x) \leq 1$

$0 \leq 2\cos(2x)+2 \leq 4$

$\therefore y = 0, \pi$

$2\cos(2x)+2=0$

$\cos(2x) = -1$

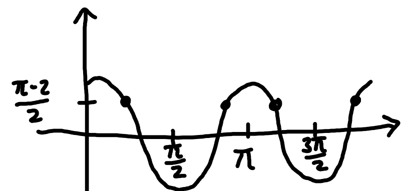
$2x = \pi, 3\pi$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

total solutions = 6

$2\cos(2x)+2=\pi$

$\cos(2x) = \frac{\pi-2}{2}$



4 solutions

F. For a real number x we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x. Let

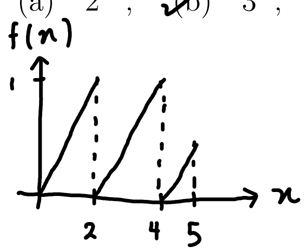
$$f(x) = \frac{x}{2} - \left\lfloor \frac{x}{2} \right\rfloor .$$

The smallest number of equal width strips for which the trapezium rule produces an **overestimate** for the integral

$$\int_0^5 f(x)\mathrm{d}x$$

is

(a) 2 , (b) 3 , (c) 4 , (d) 5, (e) it never produces an overestimate.



(Real) Area $= \frac{2\times1}{2} \times 2 + \frac{1\times0.5}{2}$

$= 2.25$

If number of strips = 2 (trapezium rule)

$\int_0^5 f(x)\,dx = \frac{1}{2} \times \frac{5}{2}\left[0+2\left(\frac{1}{4}\right)+\frac{1}{2}\right]$

$= \frac{5}{4}\left[\frac{1}{2}+\frac{1}{2}\right)$

$= \frac{5}{4} < 2.25 \quad \therefore$ underestimate

if number of strips = 3

$\int_0^5 f(x)\,dx = \frac{1}{2} \times \frac{5}{3}\left[0+2\left(\frac{5}{6}+\frac{2}{3}\right)+\frac{1}{2}\right]$

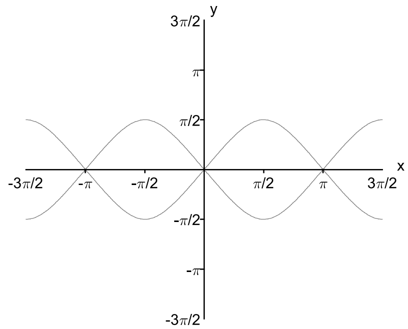
$= \frac{1}{2} \times \frac{5}{3}\left[2\left(\frac{9}{6}\right)+\frac{1}{2}\right]$

$= \frac{5}{6}\left(\frac{7}{2}\right)$

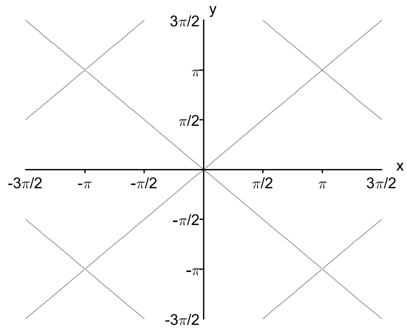
$= \frac{35}{12} > 2.25 \quad \therefore$ overestimate

$\cos^2(x) = \cos^2(y)$
$\cos(x) = \cos(-x)$
$\therefore$ when $x = -y$
$\cos^2(x) = \cos^2 y)$
$\therefore$ elimignatin a and d

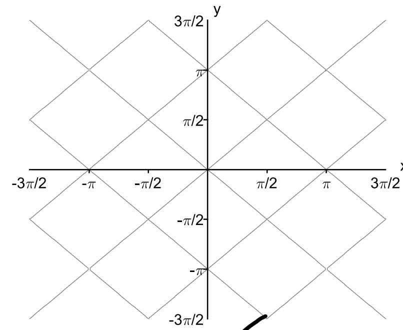
G. The graph of $\cos^2(x) = \cos^2(y)$ is sketched in



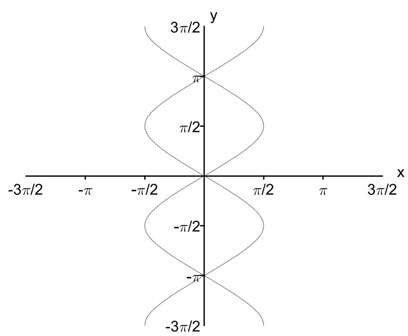
(a)



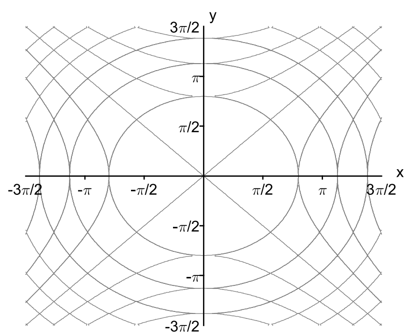
(b)



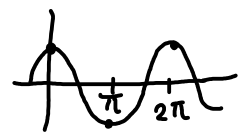
(c) ✓



(d)



(e)



when $x = 0$, $\cos x = 1$
when $y = \pi$, $\cos y = -1$
$\therefore$ $\cos^2 x$ and $\cos^2 y = 1$
$\therefore$ eliminating b and e

H. How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4 - 5x^2 - 6x^3) = 2$$

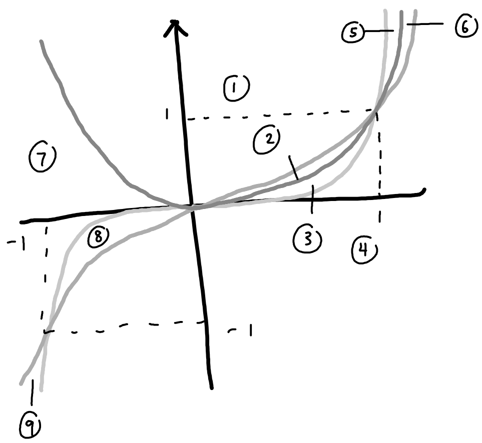
(a) None, (b) 1, (c) ✓ 2, (d) 4, (e) Infinitely many.

$\log_{x^2+2}(4 - 5x^2 - 6x^3) = 2$
$(x^2+2)^2 = 4 - 5x^2 - 6x^3$
$x^4 + 4x^2 + 4 = 4 - 5x^2 - 6x^3$
$x^4 + 9x^2 + 6x^3 = 0$
$x^2(x^2 + 6x + 9) = 0$
$x^2 (x+3)(x+3) = 0$
$x^2(x+3)^2 = 0$
$x = 0$, $x = -3$
$\therefore$ 2 distinct solutions

I. Into how many regions is the plane divided when the following equations are graphed, not considering the axes?

$$y = x^3$$
$$y = x^4$$
$$y = x^5$$

(a) 6, (b) 7, (c) 8, (d) 9, (e) 10. ✓



$x^3 = x^4$

$0 = x^3(x-1)$

points of intersection at $x=1$, $x=0$

$x^3 = x^5$

$0 = x^3(x^2-1)$

points of intersection at $x=0$, $x=1$, $x=-1$

$x^4 = x^5$

$0 = x^4(x-1)$

points of intersection at $x=0$, $x=1$

J. Which is the largest of the following numbers?

(a) $\frac{\sqrt{7}}{2}$, ✓ (b) $\frac{5}{4}$, (c) $\frac{\sqrt{10!}}{3(6!)}$, (d) $\frac{\log_2(30)}{\log_3(85)}$, (e) $\frac{1+\sqrt{6}}{3}$.

$a = \frac{\sqrt{7}}{2}$

$a^2 = \frac{7}{4}$

$= \frac{28}{16}$

$b = \frac{5}{4}$

$b^2 = \frac{25}{16}$

$\frac{28}{16} > \frac{25}{16}$ so $a^2 > b^2$

$c = \frac{\sqrt{10!}}{3(6!)} = \frac{\sqrt{10\times 8\times 7} \times \sqrt{9} \times \sqrt{6!}}{3(6!)}$

$= \frac{3\sqrt{560} \times \sqrt{6!}}{3(6!)}$

$= \frac{\sqrt{560}}{\sqrt{6!}} = \frac{\sqrt{560}}{\sqrt{720}}$

$= \frac{\sqrt{7}}{\sqrt{9}}$

$c = \frac{\sqrt{7}}{3} < \frac{\sqrt{7}}{2}$ so $c < a$

$d = \frac{\log_2(30)}{\log_3(85)}$

let $\log_2(30 = x)$

$2^x = 30$

$2^4 = 16$ $2^5 = 32$

$4 < x < 5$

let $\log_3(85) = y$

$3^y = 85$

$3^4 = 81$ $3^5 = 243$

$4 < y < 5$

so $d = \frac{x}{y}$ $\frac{4}{5} < \frac{x}{y} < \frac{5}{4}$

$\frac{5}{4} < \frac{\sqrt{7}}{2}$, $\therefore d < a$

$e = \frac{1+\sqrt{6}}{3}$

$1 < e < \frac{7}{6}$

$1 < e < \frac{\sqrt{49}}{6}$

$(2.5)^2 = 6.25$

$2 < \sqrt{6} < 2.5$

$a = \frac{\sqrt{7}}{2} = \frac{3\sqrt{7}}{6} = \frac{\sqrt{63}}{6}$

$\frac{\sqrt{63}}{6} > \frac{\sqrt{49}}{6}$

$\therefore a > e$

Turn over

2. For **ALL APPLICANTS.**

(i) Expand and simplify

$$(a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + ab^{n-1} + b^n).$$

(ii) The prime number 3 has the property that it is one less than a square number. Are there any other prime numbers with this property? Justify your answer.

(iii) Find all the prime numbers that are one more than a cube number. Justify your answer.

(iv) Is $3^{2015} - 2^{2015}$ a prime number? Explain your reasoning carefully.

(v) Is there a positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a cube number? Explain your reasoning carefully.

i) $(a-b)(a^n + a^{n-1}b + \ldots + ab^{n-1} + b^n)$

$= a^{n+1} + \cancel{a^n b} + \cancel{\ldots} + \cancel{a^2b^{n-1}} + \cancel{ab^n} - (\cancel{a^n b} + \cancel{a^{n-1}b^2} + \cancel{\ldots} + \cancel{ab^n} + b^{n+1})$

$= a^{n+1} - b^{n+1}$

ii) we want n such that n^2-1 is prime

$n^2 - 1 = (n-1)(n+1)$

we need one of $n-1$ or $n+1$ to $=1$ if n^2-1 is to be prime

as prime means a number has 2 factors, one of which is 1

as $n+1 > n-1$

$n-1=1$

this is only true when $n=2$

giving $2^2-1=3$

∴ 3 is the only prime with this property

iii) we want n such that n^3+1 is prime

$n^3+1 = (n+1)(n^2-n+1)$

to be prime either $n+1$ or $n^2-n+1 = 1$ (same reasoning as part ii)

$n+1=1$

$n=0$

but 0^3+1 is not prime ∴ not true for $n=0$

$n^2-n+1=1$

$n(n-1)=0$

$n=0 \quad n=1$

$1^3+1=2$

2 is prime and is only prime of form n^3+1

2iv. $3^{2015} - 2^{2015}$ $\quad 2015 \div 5 = 403$

$= (3^5)^{403} - (2^5)^{403}$

$\therefore$ $a = 3^5$ $\quad b = 2^5$ $\quad n+1 = 403$ $\quad$ using the formula in i.

$n = 402$

$$3^{2015} - 2^{2015} = (3^5 - 2^5)((3^5)^{402} + \dots + ((2)^5)^{402})$$

$$= (243 - 32)(3^{2010} + \dots + 2^{2010})$$

$$= 211(3^{2010} + \dots + 2^{2010})$$

Neither 211 nor $(3^{2010} + \dots + 2^{2010}) = 1$

$\therefore$ $3^{2015} - 2^{2015}$ is the product of two factors which are not 1, making $3^{2015} - 2^{2015}$ not a prime number.

v. k^3 is a cube number

$(k+1)^3$ is the next cube number

$(k+1)^3 = k^3 + 3k^2 + 3k + 1$

$k^3 < k^3 + 2k^2 + 2k + 1 < k^3 + 3k^2 + 3k + 1$ $\quad (k > 0)$

$(k$ is an integer$)$

$\therefore$ $k^3 + 2k^2 + 2k + 1$ is between two cube numbers, so there is no positive integer k where $k^3 + 2k^2 + 2k + 1$ is a cube number.

3.

For **APPLICANTS IN** $\left\{ \begin{array}{l} \textbf{MATHEMATICS} \\ \textbf{MATHEMATICS \& STATISTICS} \\ \textbf{MATHEMATICS \& PHILOSOPHY} \\ \textbf{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$ **ONLY.**

Computer Science and *Computer Science & Philosophy* **applicants should turn to page 16.**

In this question we shall investigate when functions are close approximations to each other. We define $|x|$ to be equal to x if $x \geqslant 0$ and to $-x$ if $x < 0$. With this notation we say that a function f is an **excellent approximation** to a function g if

$$|f(x) - g(x)| \leqslant \frac{1}{320} \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2};$$

we say that f is a **good approximation** to a function g if

$$|f(x) - g(x)| \leqslant \frac{1}{100} \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2}.$$

For example, any function f is an excellent approximation to itself. If f is an excellent approximation to g then f is certainly a good approximation to g, but the converse need not hold.

(i) Give an example of two functions f and g such that f is a good approximation to g but f is *not* an excellent approximation to g.

(ii) Show that if

$$f(x) = x \qquad \text{and} \qquad g(x) = x + \frac{\sin(4x^2)}{400}$$

then f is an excellent approximation to g.

For the remainder of the question we are going to a try to find a good approximation to the exponential function. This function, which we shall call h, satisfies the following equation

$$h(x) = 1 + \int_0^x h(t)\mathrm{d}t \qquad \text{whenever} \qquad x \geqslant 0.$$

You may not use any other properties of the exponential function during this question, and any attempt to do so will receive no marks.

Let

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

(iii) Show that if

$$g(x) = 1 + \int_0^x f(t)\mathrm{d}t,$$

then f is an excellent approximation to g.

(iv) Show that for $x \geqslant 0$

$$h(x) - f(x) = g(x) - f(x) + \int_0^x (h(t) - f(t))\mathrm{d}t.$$

(v) You are given that $h(x) - f(x)$ has a maximum value on the interval $0 \leqslant x \leqslant 1/2$ at $x = x_0$. Explain why

$$\int_0^x (h(t) - f(t))\mathrm{d}t \leqslant \frac{1}{2}(h(x_0) - f(x_0)) \qquad \text{whenever} \qquad 0 \leqslant x \leqslant \frac{1}{2}.$$

(vi) You are also given that $f(x) \leqslant h(x)$ for all $0 \leqslant x \leqslant \frac{1}{2}$. Show that f is a good approximation to h when $0 \leqslant x \leqslant \frac{1}{2}$.

3i. $f(x) = x + \frac{1}{100}$ $\qquad g(x) = x$

$$|f(x) - g(x)| = \frac{1}{100}$$

$$\frac{1}{100} = \frac{1}{100} > \frac{1}{320}$$

ii. $f(x) - g(x) = x - \left(x + \frac{\sin(4x^2)}{400}\right)$

$$|f(x) - g(x)| = \left|-\frac{\sin(4x^2)}{400}\right|$$

$$-1 \leq \sin(4x^2) \leq 1$$

$$0 \leq |\sin(4x^2)| \leq 1$$

$$0 \leq |f(x) - g(x)| \leq \frac{1}{400}$$

$$\frac{1}{400} < \frac{1}{320}$$

$\therefore$ excellent approximation

iii. $h(x) = 1 + \int_0^x h(t)\,dt$ $\qquad g(x) = 1 + \int_0^x f(t)\,dt$

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$g(x) = 1 + \int_0^x 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\,dt$$

$$g(x) = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)$$

$$f(x) - g(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right)$$

$$|f(x) - g(x)| = \left|-\frac{x^4}{24}\right| = \frac{x^4}{24}$$

$$0 \leq x \leq \frac{1}{2}$$

$$0 \leq \frac{x^4}{24} \leq \frac{1}{384}$$

$$\frac{1}{384} < \frac{1}{320}$$

$\therefore$ excellent approximation

3iv. $h(x) - f(x) = 1 + \int_0^x h(t)\,dt - f(x)$

$1 = g(x) - \int_0^x f(t)\,dt$

$h(x) - f(x) = g(x) - \int_0^x f(t)\,dt + \int_0^x h(t)\,dt - f(x)$

$= g(x) - f(x) + \int_0^x h(t) - f(t)\,dt$

v. $\int_0^x (h(t) - f(t))\,dt \leq (x-0)\left[h(x_0) - f(x_0)\right]$ $\quad 0 \leq x \leq \frac{1}{2}$

max value of $x = \frac{1}{2}$

$\int_0^x (h(t) - f(t)\,dt \leq \frac{1}{2}\left(h(x_0) - f(x_0)\right)$

vi. $h(x_0) - f(x_0) = g(x_0) - f(x_0) + \int_0^{x_0} h(t) - f(t)\,dt$

$h(x_0) - g(x_0) \leq \frac{1}{2}(h(x_0) - f(x_0))$

$2h(x_0) - 2g(x_0) \leq h(x_0) - f(x_0)$

$h(x_0) \leq 2g(x_0) - f(x_0)$

$h(x_0) - f(x_0) \leq 2(g(x_0) - f(x_0))$

$h(x) \geq f(x_0)$

$g(x_0) - f(x_0) = \frac{x^4}{24}$

(part iii)

$|h(x_0) - f(x_0)| \leq \frac{2x_0^4}{24}$

$|h(x_0) - f(x_0)| \leq \frac{x_0^4}{12}$ $\quad 0 \leq \frac{x^4}{12} \leq \frac{1}{192}$ $\quad 0 \leq x \leq \frac{1}{2}$

$\frac{1}{100} > \frac{1}{192} > \frac{1}{320}$

$\therefore f$ is a good approximation to h when $0 \leq x \leq \frac{1}{2}$

4.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{MATHEMATICS}\\ \textbf{MATHEMATICS \& STATISTICS}\\ \textbf{MATHEMATICS \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

Mathematics & Computer Science, *Computer Science* and *Computer Science & Philosophy* **applicants should turn to page 16.**

A circle A passes through the points $(-1, 0)$ and $(1, 0)$. Circle A has centre (m, h), and radius r.

(i) Determine m and write r in terms of h.

(ii) Given a third point (x_0, y_0) and $y_0 \neq 0$ show that there is a unique circle passing through the three points $(-1, 0)$, $(1, 0)$, (x_0, y_0).

For the remainder of the question we consider three circles A, B, and C, each passing through the points $(-1, 0)$, $(1, 0)$. Each circle is cut into regions by the other two circles. For a group of three such circles, we will say the **lopsidedness** of a circle is the fraction of the full area of that circle taken by its largest region.

(iii) Let circle A additionally pass through the point $(1, 2)$, circle B pass through $(0, 1)$, and let circle C pass through the point $(0, -4)$. What is the lopsidedness of circle A?

(iv) Let $p > 0$. Now let A pass through $(1, 2p)$, B pass through $(0, 1)$, and C pass through $(-1, -2p)$. Show that the value of p minimising the lopsidedness of circle B satisfies the equation

$$(p^2 + 1)\tan^{-1}\left(\frac{1}{p}\right) - p = \frac{\pi}{6}.$$

Note that $\tan^{-1}(x)$ is sometimes written as $\arctan(x)$ and is the value of θ in the range $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ such that $\tan(\theta) = x$.

i) centre : (m, h)

$(x-m)^2 + (y-h)^2 = r^2$

using $(1,0)$:

$(1-m)^2 + h^2 = r^2$

using $(-1,0)$:

$(-1-m)^2 + h^2 = r^2$

$(1-m)^2 = (-1-m)^2$

$\therefore\ m = 0$

$r^2 = 1 + h^2$

$r = \sqrt{1+h^2}$

ii) using x_0, y_0

$(x_0 - m)^2 + (y_0 - h)^2 = r^2$

by $m=0$ and $r^2 = 1+h^2$

$(x_0)^2 + (y_0)^2 - 2hy_0 + h^2 = 1 - h^2$

$(x_0)^2 + (y_0)^2 = 2hy_0 + 1$

$h = \sqrt{r^2 - 1} \longrightarrow$ from part i)

$r^2 = 1 + h^2$

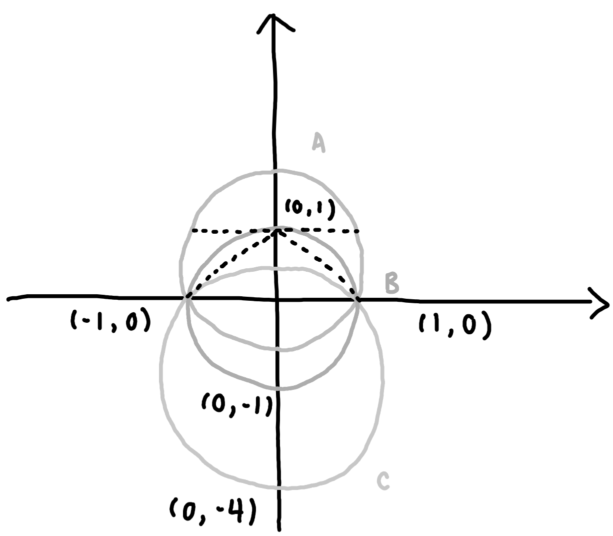
$h^2 = r^2 - 1$

$h = \sqrt{r^2-1}$

$x_0^2 + y_0^2 = 1 + 2y_0\sqrt{r^2-1}$

$\therefore$ depends on x_0 and y_0

iii)



$x_0^2 + y_0^2 = 1 + 2y_0 h$

$1 + 4 = 1 + 4h$

$h = 1 \quad \therefore r = \sqrt{2}$

$\therefore$ centre of $A = (0,1)$

area of $A = \pi r^2$

$= 2\pi$

area of triangle from diagram

$= \frac{1}{2} \times$ base $\times$ height $= 1$

and area of semicircle of $B = \frac{\pi}{2}$

small area of A under axis = semicircle area − triangle area

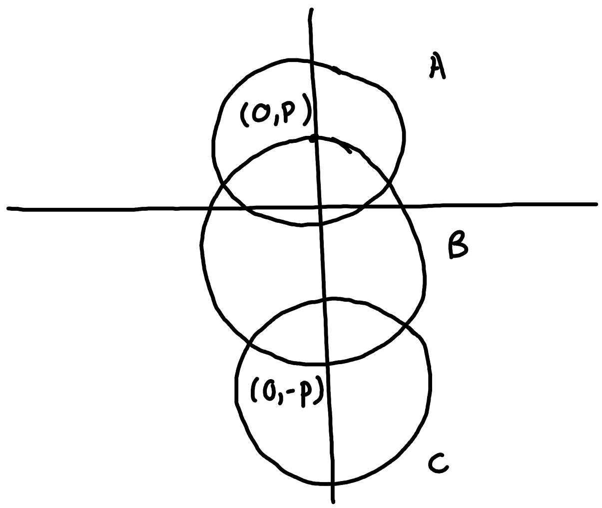
$= \frac{\pi}{2} - 1$

$$\frac{2\pi\left(\frac{\pi}{2} - 1\right) - \frac{\pi}{2}}{2\pi} = L$$

$$L = \frac{\pi + 1}{2\pi}$$

4iv) centre for A now at $(0, p)$

centre for C at $(0, -p)$



lopsidedness at min when all areas are equal

($\frac{\pi}{3}$ is the area)

y axis cuts the middle region into 2

so difference between a sector and triangle should be $\frac{\pi}{6}$

radius is now $\sqrt{1+p^2}$

$\therefore$ area of sector $= \frac{1}{2} r^2 \theta$

$$\theta = \arctan\left(\frac{1}{p}\right)$$

$\therefore$ 2 sector areas $(1+p^2)\arctan\left(\frac{1}{p}\right)$

2 × triangle area $= 2 \times \frac{1}{2} \times p \times 1 = p$

$$\therefore (1+p^2)\arctan\left(\frac{1}{p}\right) - p = \frac{\pi}{6}$$

5. For **ALL APPLICANTS.**

The following functions are defined for all integers a, b and c:

$$\begin{aligned} p(x) &= x+1 \\ m(x) &= x-1 \\ s(x,y,z) &= \begin{cases} y & \text{if } x \leqslant 0 \\ z & \text{if } x > 0. \end{cases} \end{aligned}$$

(i) Show that the value of

$$s\Big(\, s(p(0), m(0), m(m(0))),\ \ s(p(0), m(0), p(p(0))),\ \ s(m(0), p(0), m(p(0)))\, \Big)$$

is 2.

Let f be a function defined, for all integers a and b, as follows:

$$f(a,b) = s(\,b,\ p(a),\ p(f(a, m(b)))\,).$$

(ii) What is the value of $f(5,2)$?

(iii) Give a simple formula for the value of $f(a,b)$ for all integers a and all *positive* integers b, and explain why this formula holds.

(iv) Define a function $g(a,b)$ in a similar way to f, using only the functions p, m and s, so that the value of $g(a,b)$ is equal to the sum of a and b for all integers a and all integers $b \leqslant 0$.

Explain briefly why your function gives the correct value for all such values of a and b.

5i. $p(0) = 0+1 = 1$ $m(0) = 0-1 = -1$ $m(m(0)) = m(-1) = -1-1 = -2$

$p(p(0)) = p(1) = 1+1 = 2$ $m(p(0)) = m(1) = 1-1 = 0$

$s(s(1,-1,-2), s(1,-1,2), s(-1,1,0))$

$= s(-2, 2, 1)$ $-2 < 0$

$= 2$

ii. $f(5,2) = s(2, p(5), p(f(5, m(2)))) = p(f(5,1))$

$f(5,1) = s(1, p(5), p(f(5, m(1)))) = p(f(5,0))$

$f(5,0) = s(0, p(5), p(f(5, m(0)))) = p(5)$

$f(5,2) = p(p(p(5)))$

$= 5+1+1+1$

$= 8$

iii. $f(a,b) = \underbrace{p(p(p(}_{}\cdots(p(a))),$

where the number of $p = b+1$

$f(a,b) = 1 \times (b+1) + a$

$= a+b+1$

While b is positive, the value of s taken is $p(f(a, m(b)))$, which adds 1 to the answer and reduces b by 1.

Once $b = 0$, $p(a) = 1+a$ is added to the result, leading to $1 \times b + 1 + a = a+b+1$

5 iv. $g(a,b) = s(p(b), m(g(a, p(b))), a)$

While b is negative, the value of s taken is $m(g(a, p(b)))$, which takes away 1 from the answer and increases b by 1.

This would occur $|b|$ times, until b is 0.

Then $p(b) = 1$, $1 > 0$, so a would be added.

$|b| \times -1 = b \quad (b \leq 0)$

$b + a = g(a,b)$

PMT
•resources•tuition•courses

6.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{MATHEMATICS \& COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

The world is divided into two species, vampires and werewolves. Vampires always tell the truth when talking about a vampire, but always lie when talking about a werewolf. Werewolves always tell the truth when talking about a werewolf, but always lie when talking about a vampire. (Note that this does not imply that creatures necessarily lie when speaking *to* creatures of the other species. Note also that "Zaccaria is a vampire" is a statement about Zaccaria, rather than necessarily about a vampire.)

These facts are well known to both sides, and creatures can tell instinctively which species an individual belongs to.

In your answers to the questions below, you may abbreviate "vampire" and "werewolf" to "V" and "W", respectively.

(i) Azrael says, "Beela is a werewolf." Explain why Azrael must be a werewolf, but that we cannot tell anything about Beela.

(ii) Cesare says, "Dita says 'Elith is a vampire.'" What can we infer about any of the three from this statement? Explain your answer.

(iii) Suppose N creatures (where $N \geq 2$) are sitting around a circular table. Each tells their right-hand neighbour, "You lie about your right-hand neighbour." What can we infer about N? What can we infer about the arrangement of creatures around the table? Explain your answer.

(iv) Consider a similar situation to that in part (iii) (possibly for a different value of N), except that now each tells their right-hand neighbour, "Your right-hand neighbour lies about their right-hand neighbour." Again, what can we infer about N and the arrangement of creatures around the table? Explain your answer.

6i. If B is a W, A's statement is true, making the speaker (A) a W. (Only W tell truth about W)

If B is a V, A's statement is false, making the speaker (A) a W. (Only W lie about V)

∴ A is a W, B could be either a W or a V.

ii. If C is telling the truth:
Following logic in part i, D is a vampire, E is unknown and C is also a V.

If C is lying:
D is saying "E is a W" instead of "E is a V".
This makes D a werewolf, E unknown and C, since they are lying about D, a W, a vampire.

∴ C is always a V, D and E are unknown.

iii. $\overset{1}{X}\,\overset{2}{Y}\,\overset{3}{Z}$

X says Y lies about Z

if X is telling the truth:
Y must lie, making Y and Z different species, while X and Y are the same (X tells the truth about Y).

if X is lying:
Y is telling the truth, making Y and Z the same species, while X is the other.

if X is telling the truth:

WWV (X Y Z) VVW (X Y Z)

if X is lying:

WVV (X Y Z) VWW (X Y Z)

In both cases, X and Z (the first and third) are different species.

∴ the pattern is WWVV|WWVV|..., starting at any point where N is divisible by 4

6iv. $\overset{1}{W}\ \overset{2}{X}\ \overset{3}{Y}\ \overset{4}{Z}$

W says to X that Y is lying about Z
= W says that Y is lying about Z

Following on from part iii:

If W is telling the truth:

W and Y are the same species, Z is a different one.

W_WV_VW V_VW_WV

The blank spaces can be filled in with a species different to their neighbours.

Resulting in a pattern of WVWVW...., starting in any place, where N is divisible by 2.

If ~~Z~~W is lying:

W is one species, Y and Z are a different one.

$V\_{2}\ W\ W\ \_{1}\ V\ V\ \_$

W says that $\_{1}$ is lying to V

$\_{1}$ can be either V or W.

$\_{2}$ is the opposite.

∴ there is a pattern of VWWWVV|VWWWVV|V...., starting in any place, where N is divisible by 6.

∴ There are two possible arrangements.

7.

For **APPLICANTS IN** $\left\{\begin{array}{l}\textbf{COMPUTER SCIENCE}\\ \textbf{COMPUTER SCIENCE \& PHILOSOPHY}\end{array}\right\}$ **ONLY.**

In this question we will study a mechanism for producing a set of **words**. We will only consider words containing the letters a and/or b, and that have length at least 1. We will make use of **variables**, which we shall write as capital letters, including a special start variable called S. We will also use **rules**, which show how a variable can be replaced by a sequence of variables and/or letters. Starting with the start variable S, we repeatedly replace one of the variables according to one of the rules (in any order) until no variables remain.

For example suppose the rules are

$$S \to AB, \qquad A \to AA, \qquad A \to a, \qquad B \to bb.$$

We can produce the word $aabb$ as follows; at each point, the variable that is replaced is underlined:

$$\underline{S} \to \underline{A}B \to A\underline{A}B \to \underline{A}aB \to aa\underline{B} \to aabb.$$

(i) Show that the above rules can be used to produce all words of the form $a^n bb$ with $n \geq 1$, where a^n represents n consecutive a's.

Also briefly explain why the rules can be used to produce no other words.

(ii) Give a precise description of the words produced by the following rules.

$$S \to ab, \qquad S \to aSb.$$

(iii) A **palindrome** is a word that reads the same forwards as backwards, for example $bbaabb$. Give rules that produce all palindromes (and no other words).

(iv) Consider the words with the same number of a's as b's; for example, $aababb$. Write down rules that produce these words (and no others).

(v) Suppose you are given a collection of rules that produces the words in L_1, and another collection of rules that produces the words in L_2. Show how to produce a single set of rules that produce all words in L_1 or L_2, or both (and no other words). Hint: you may introduce new variables if you want.

7) i)

Rule Used	Result
S	AB
A	AA OR a
B	bb

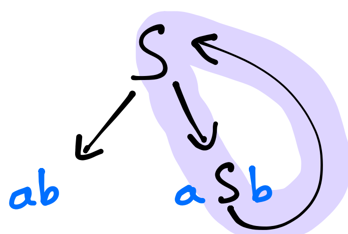
So, to compose words of the form $a^n bb$ from S

Every A will end up producing at least 1 a ($n \geq 1$). Since we can replace A with AA as many times as we want, there is no limit on the # of a's in our word

b's only come in pairs because of how the rule for B works. So we can always end our word in bb if we use only 1 B at the very end (right-most)

∴ Words of the form $a^n b$ can be produced from the given rules

ii)



example words:

$S \rightarrow ab$

$S \rightarrow aSb \rightarrow a(ab)b \equiv a^2 b^2$

$S \rightarrow aSb \rightarrow a(aSb)b \rightarrow a^2(ab)b^2 \equiv a^3 b^3$

Whenever we have an S, we can either replace it with ab, or with a new S sandwiched between an a and b. A word is formed when we choose $S \rightarrow ab$ and break the cycle.

⇒ Our words will ALWAYS look like a consecutive series of a's followed by a consecutive series of b's (same length as we ONLY get ab pairs)

$= a^n b^n$

iii) Since we can only produce palindromes, all of our rules should produce palindromic / symmetric, or else there is a chance of producing a word that is not a palindrome.

Example palindromes: a, b, aa, bb, aba, bab, aabaa, bbabb

Need a rule to output single letters: $S \rightarrow a$, $S \rightarrow b$

Need a rule to output doubles: $S \rightarrow aa$, $S \rightarrow bb$

Need a sandwich rule, could leverage existing rules $S \rightarrow aSa$, $S \rightarrow bSb$

All palindromes can be constructed these 6 rules.

iv) All rules must output the same number of a's and b's or else there is the possibility of a word with an unequal number of a's and b's

$\Rightarrow$ $S \to ab$, $S \to ba$

Just like in (iii), we can sandwich rules within eachother to make longer words where # a's = # b's

$\Rightarrow$ $S \to aSb$, $S \to bSa$

Additionally, we can have S call just itself twice (this keeps # a's = # b's)

$\Rightarrow$ $S \to SS$

v) The rules that produce the words in L_1 :

$$\left\{ \begin{array}{l} S_1 \to \dots , \\ S_1 \to \dots , \\ \dots \to \dots , \\ \dots \to \dots , \\ \vdots \end{array} \right\}$$

The rules that produce the words in L :

$$\left\{ \begin{array}{l} S_2 \to \dots , \\ S \to \dots , \\ \dots \to \dots , \\ \dots \to \dots , \\ \vdots \end{array} \right\}$$

Since all words and rules in L_1 can be accessed via S_1 (starting rule for L_1) and similarly for S_2 and L_2,

The set of rules : $S \to S_1$, $S \to S_2$ and all rules in L_1 and L_2 are sufficient to produce all words in L_1 and L_2 .